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GRAVITATIONAL OSCILLATIONS IN A ROTATING PARABOLOIDAL BASIN: A CLASSICAL PROBLEM REVISITED

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ABSTRACT

The effect of ellipticity of the free surface on the frequencies of axisymmetric normal gravity modes in a rotating shallow layer of liquid contained in a circular cylinder with paraboloidal bottom is investigated. The classical shallow-water theory due to Lord Kelvin for the flat bottom case neglects the curvature of the free surface due to rotation. In 1962 Fultz and Platzman treated the flat-bottom case, taking into account the effect of this ellipticity. The present work is similar to that of Fultz and Platzman, but is more general and treats the flat bottom as a special case.

Suitable nondimensional parameters are defined, namely a depth parameter, a rotation parameter, and a frequency parameter. The problem is formulated in terms of cylindrical polar coordinates. It is a Sturm-Liouville problem and the differential operator is self-adjoint. The problem is solved by the Galerkin method, but the solution thus obtained is not in a closed form. The problem is also solved in a closed form by making use of the Legendre functions, but since these functions are not tabulated very extensively the solution involving the Galerkin method is used for computational purposes. A discussion of these Legendre function solutions is given and the results are compared with those of the Galerkin method whenever possible.

Poincare treated the gravity modes in a parabola without a cylinder and not taking the ellipticity into account. The writers treated this case including the effect of ellipticity and obtained results that differ radically from those in which ellipticity is neglected. In particular, according to the present theory, rotation has no effect on the frequency of the fundamental mode.

Key-words: Gravity modes, ellipticity, Fultz & Platzman, Galerkin method.

INTRODUCTION

Laplace classified the tidal phenomena into zonal and tesseral oscillations. In the zonal oscillations, the surface of the liquid is divided into annular zones symmetric with respect to the vertical axis through the center. In the tesseral oscillations, the surface of the liquid is divided into a number of compartments by nodal circles and nodal diameters. In the tesseral oscillations Hough (1898), Lamb (1932), Proudman (1913, 1953), Sverdrup (1927), Rayleigh (1876) and Taylor (1922) distinguished between oscillations of the first class and oscillations of the second class on the following basis. If σ is the frequency of oscillation and ω is the frequency of rotation, then oscilla-

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tions of the first class are those for which $\sigma \longrightarrow \sigma_0 (\neq 0)$ as $\omega \longrightarrow 0$ and oscilations of the second class are those for which $\sigma \longrightarrow 0(\omega)$ as $\omega \rightarrow 0$. Bjerknes, Bjerknes, Solberg and Bergeron (1934) distinguished between these two types of oscillation by means of the ratio $\sigma/2\omega$. Gravity modes (oscillations of the first class) are those for which $\frac{\sigma}{2\omega} \geq 1$. Elastoid-inertia modes (oscillations of the second class) are those for which $\frac{\sigma}{2\omega} \leq 1$. For the gravity modes, gravity appears in the frequency equation. In the case of the rotational (elastoidinertia) modes, the frequency for a given mode is a function mainly of the ratio of the depth of the liquid to the radius of the container and gravity does not play an important role in the frequency equation.

In this paper, oscillations of the zonal type alone are considered; moreover, the discussion is restricted to gravity modes. In the mathematical analysis, the latter restriction is imposed by introducing the approximations of the shallow-water theory, called by V. Bjerknes the quasi-static approximation. This means that in each vertical, equilibrium exists not only before the motion but also during the motion, vertical accelerations of the liquid being considered negligible compared to gravity.

It is a known fact from mechanics that the effect of rotation is to stabilize the system. For example, if a loose chain is wound round a motor and the motor is run fast for some time, then if the chain is slipped off the motor, the chain rolls as a rigid body. (This inertial stability associated with rotation is necessary for the elastoid-inertia oscillations.) In this paper, the question to be answered is this: what is the effect of rotation on gravity waves? Thomson (1879) tried to answer this question. His main interest then was to study the tidal motions on the rotating earth. He considered a shallow layer of water in a circular cylinder with flat bottom and made the quasi-static approximation to the pressure field. Lord Kelvin (Thomson) considered small rotations and neglected the ellipticity of the free surface due to rotation. If σ is the frequency of the rotating mode, σ_0 is the frequency of the nonrotating mode and ω is the rotation frequency then the result he obtained is

$$\sigma^2 = \sigma_0{}^2 + 4\omega^2 \tag{1.1}$$

This result shows that rotation increases the frequency and thus increases the restoring tendency of the system. However, if the ellipticity is taken into account, this is not always true, especially for the higher modes, as will be seen later in this study.

Poisson treated the same problem for arbitrary depth, but at that time the theory of Bessel functions was not worked out. Later Rayleigh (1876) worked out the same case again making use of the Bessel functions. Poincare (1910) treated the gravity modes in a parabola without taking into account the ellipticity of the free surface. Hidaka (1931) considered the case of gravity modes in a shallow cone without taking into account the ellipticity.

Fultz and Platzman (1962) considered the flat-bottom case again but taking the ellipticity into account. Their results are different from those of the classical shallow-water theory and they clearly showed that, however small the rotation may be, still the ellipticity effects the value of the coefficient which multiplies ω^2 in (1.1). They defined the following nondimensional parameters :

frequency	$\beta \equiv \sigma \; a/\sqrt{gH}$	(1.2)
a rotation parameter	$\alpha \equiv 4\omega^2 a^2/gH$	(1.3)
a frequency parameter	$K \equiv \dfrac{\sigma^2-\sigma_o^2}{4\omega^2} = \dfrac{\beta^2-\beta_o^2}{\alpha}$	(1.4)

Here a is the radius of the cylinder and H is a depth parameter defined by

$$H \equiv V/\pi a^2 \qquad (1.5)$$

where V is the volume of the liquid.

From (1.1) the classical theory gives $K = 1$ for all α for any mode. Fultz and Platmzman gave the following values of K in the limit $\alpha \to 0$ for the first three modes.

$$\lim_{\alpha \to 0} K_1 = 0.6941$$

$$\lim_{\alpha \to 0} K_2 = -0.0254$$

$$\lim_{\alpha \to 0} K_3 = -1.1563$$

For values of $\alpha > 0$, their analysis shows that K is a monotone decreasing function of α. These values differ markedly from the result $K = 1$ given by classical shallow-water theory.

In our opinion, the work of Fultz and Platzman (1962) is the first systematic attempt to study the complete effect of rotation on lhese gravily modes. We consider here a more general case, namely a circular cylinder with paraboloidal bottom. A suitable depth parameter δ is defined as

$$\delta \equiv \frac{2a^2}{fH} \qquad (1.6)$$

where f is the focal length of the parabola. Using (1.5), this becomes

$$\delta = \frac{2\pi a^4}{fV} \qquad (1.7)$$

2. FORMULATION AND THE METHOD OF SOLUTION OF THE GENERAL PROBLEM

In this section, the gravity modes in a circular cylinder with any law of depth are considered and a method of solution is outlined. This theory is applicable to gravity modes in a uniform gravitational field of absolute gravity.

Let x, y = Cartesian coordinates in the horizontal plane

u, v = velocity components in the x, y directions, respectively

t = time

g = acceleration due to gravity

η = elevation of the free surface above the undisturbed level.

The linearized relative equations of motion, and continuity equation, in the Cartesian system for the rotating case are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= \frac{1}{\rho'}\frac{\partial P}{\partial x} \\ \frac{\partial v}{\partial t} + 2\omega u &= \frac{1}{\rho'}\frac{\partial P}{\partial y} \end{aligned}\right\} \quad (2.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2.2)$$

where P is the pressure and ρ' is the density.

Making the hydrostatic approximation to the pressure field (2.1) and (2.2) become

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v}{\partial t} + 2\omega u &= -g\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0 \end{aligned}\right\} \quad (2.3)$$

where h is the depth of the liquid.

Assuming the motion to be simple harmonic with a comple time factor $e^{i(\sigma t+\epsilon)}$, we get

$$\left.\begin{aligned} i\sigma u - 2\omega v &= -g\frac{\partial \eta}{\partial x} \\ i\sigma v + 2\omega u &= -g\frac{\partial \eta}{\partial y} \end{aligned}\right\} \quad (2.4)$$

$$i\sigma\eta = -\frac{\partial}{\partial x}(hu) - \frac{\partial}{\partial y}(hv) \quad (2.5)$$

Solving for u and v from (2.4) we get

$$\left.\begin{aligned} u &= \frac{g}{(\sigma^2-4\omega^2)}\left(i\sigma\frac{\partial\eta}{\partial x}+2\omega\frac{\partial\eta}{\partial y}\right) \\ v &= \frac{g}{(\sigma^2-4\omega^2)}\left(i\sigma\frac{\partial\eta}{\partial y}-2\omega\frac{\partial\eta}{\partial x}\right) \end{aligned}\right\} \quad (2.6)$$

Substituting (2.6) in (2.5) and rearranging the terms we get

$$gh\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)+g\left(\frac{\partial h}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial\eta}{\partial y}\right)+(\sigma^2-4\omega^2)\eta$$

$$+\frac{2\omega}{i\sigma}\left(\frac{\partial h}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial h}{\partial y}\frac{\partial\eta}{\partial x}\right)=0 \qquad (2.7)$$

Transforming into polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$ we have the results:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\ &\frac{\partial h}{\partial x}\frac{\partial}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial}{\partial y}=\frac{\partial h}{\partial r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial h}{\partial\theta}\frac{\partial}{\partial\theta} \\ &\frac{\partial h}{\partial x}\frac{\partial}{\partial y}-\frac{\partial h}{\partial y}\frac{\partial}{\partial x}=\frac{1}{r}\left(\frac{\partial h}{\partial r}\frac{\partial}{\partial\theta}-\frac{\partial h}{\partial\theta}\frac{\partial}{\partial r}\right) \end{aligned}\right\} \quad (2.8)$$

Substituting (2.8) in (2.7) we get

$$gh\left(\frac{\partial^2\eta}{\partial r^2}+\frac{1}{r}\frac{\partial n}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\eta}{\partial\theta^2}\Big)+(\sigma^2-4\omega^2)\eta$$

$$+g\left(\frac{\partial h}{\partial r}\frac{\partial\eta}{\partial r}+\frac{1}{r^2}\frac{\partial h}{\partial\theta}\frac{\partial\eta}{\partial\theta}\right)+\frac{2\omega}{i\sigma r}\left(\frac{\partial h}{\partial r}\frac{\partial\eta}{\partial\theta}-\frac{\partial h}{\partial\theta}\frac{\partial\eta}{\partial r}\right) \qquad (2.9)$$

Since we are concerned with axisymmetric modes only, $\frac{\partial}{\partial\theta}=0$ and (2.9) reduces to

$$gh\left(\frac{d^2\eta}{dr^2}+\frac{1}{r}\frac{d\eta}{dr}\right)+g\frac{dh}{dr}\frac{d\eta}{dr}+(\sigma^2-4\omega^2)\eta=0 \qquad (2.10)$$

Through multiplying by a^2/gH we get from (2.10)

$$h^*\left(\frac{d^2\eta}{d\rho^2}+\frac{1}{\rho}\frac{d\eta}{d\rho}\right)+\frac{dh^*}{d\rho}\frac{d\eta}{d\rho}+(\beta^2-\alpha)\eta=0 \qquad (2.11)$$

$$h^* \equiv h/H, \qquad \rho \equiv \frac{r}{a}$$

Here H is defined by (1.5) and h* is a dimensionless depth, while β and α are the dimensionless quantities defined in (1.2) and (1.3).

Equation (2.11) applies for any law of depth variation — that is, for any function $h^*(\rho)$ — and can be written as

$$(h^* \rho \eta')' + (\beta^2 - \alpha)\, \rho \eta = 0 \qquad (2.12)$$

where the prime denotes differentiation with respect to ρ. Equation (2.12) is self-adjoint and it is in the standard form of the Sturm-Liouville equation. The following are the boundary conditions:

$$\left.\frac{d\eta}{d\rho}\right)_{\rho=1} = 0 \qquad (2.13)$$

that is, the velocity normal to the wall is zero; and

$$\eta)_{\rho=0} = \text{finite} \qquad (2.14)$$

For axisymmetric modes the center boundary condition is $\left(\frac{\partial\eta}{\partial\rho}\right)_{\rho=0} = 0.$

Equation (2.12) should be solved for $\beta^2 - \alpha$ using the boundary conditions given by (2.13) and (2.14). The problem can be solved by the Galerkin method as follows.

Let $f_n(\rho)$, $n = 1, 2, 3 \cdots$ be functions orthonormal with the weight function ρ in the interval $0 \leq \rho \leq 1$. Then

$$\int_0^1 f_n f_m \rho d\rho = \delta_{m,n} \qquad (2.15)$$

where δ is the Kronecker delta. Assume

$$\eta = \sum_{n=1}^{\infty} a_n f_n(\rho) \qquad (2.16)$$

Substitute for η in (2.12), multiply by $f_m(\rho)$ and integrate with respect to ρ from 0 to 1. The result can be written in the form

$$\sum_{n=1}^{\infty} \left\{ (\beta^2 - \alpha)\, \delta_{m,n} - (m/n) \right\} = 0 \qquad (2.17)$$

$$(m,n) = \int_0^1 h^* f'_m f'_n \rho d\rho$$

Here (m/n) represents the off-diagonal elements of the matrix and (n/n) represents the diagonal elements.

Equation (2.17) represents a set of homogeneous simultaneous equations for the expansion coefficients a_n, and has a nontrivial solution only if the coefficient determinant vanishes. The latter conditions determines the set of proper values of $(\beta^2 - \alpha)$. In (2.17), the matrix elements (m/n) are determined, in principle, by the depth distribution h^* and by the functions f_n. It can be shown that self-adjoint problems can be treated by the variational method and that the solution obtained is precisely the same as that for the Galerkin method. (However, the Galerkin method works for non-self-adjoint cases also.)

The determination of $\beta^2 - \alpha$ from (2.17) is based upon a method of successive approximations. Let us assume that the functions f_n are arranged in ascending sequence of number of zeros; in fact, we may regard n as equal to the number of zeros of f_n. If the infinite matrix of (2.17) is truncated to a N x N matrix consisting of the first N rows and columns, we shall find a characteristic equation for $\beta^2 - \alpha$ having N roots. Taking first $N = 1$, we get $\beta^2 - \alpha$ for the first mode under the first approximation. Solving a 2 x 2 matrix we get two values for $\beta^2 - \alpha$ the lowest value gives the frequency of the first mode under the second approximation, and the other value gives the frequency of the second mode under the first approximation. By solving an N x N matrix, we get N values for $\beta^2 - \alpha$. If these are arranged in increasing order of magnitude, the first value gives the first mode under the Nth approximation, the second value gives the second mode under the (N-1)th approximation and the last value gives the Nth mode under the first approximation. In principle, one should make the ∞th approximation, since infinite modes exist, but in practice after some approximations, the values for two successive approximations will be close to the accuracy needed.

In this section, the scheme for the solution of the general problem, namely circular cylinder with any law of depth, is given. If the solution for a particular case is needed, the $f_n(\rho)$ functions should be so chosen that they satisfy the boundary conditions of that problem. The function h^* also is different in different cases.

3. FORMULATION OF THE PARABOLA-CYLINDER PROBLEM

In section 2, the formulation and a scheme for the solution of the general problem (circular cylinder with arbitrary law of depth) are given. In this chapter, the parabola cylinder problem (circular cylinder with paraboloidal law of depth) will be formulated and the law of depth for the various cases will be derived.

The depth parameter δ introduced in (1.6) is equal to $2\pi a^4/fV$ by (1.7). For a given parabola-cylinder combination, a and f are constant. One may therefore regard the parameter δ as representing the volume V. Let us roughly fix the range of numerical values of δ. If we substitute $V = \pi a^4/8f$ in (1.7) we get $\delta = 16$. This is the case of a circular cylinder with paraboloidal bottom (concave to the liquid, $f > 0$) in which the liquid completely fills the parabola and reaches the bottom of the cylinder (figure 1b). For

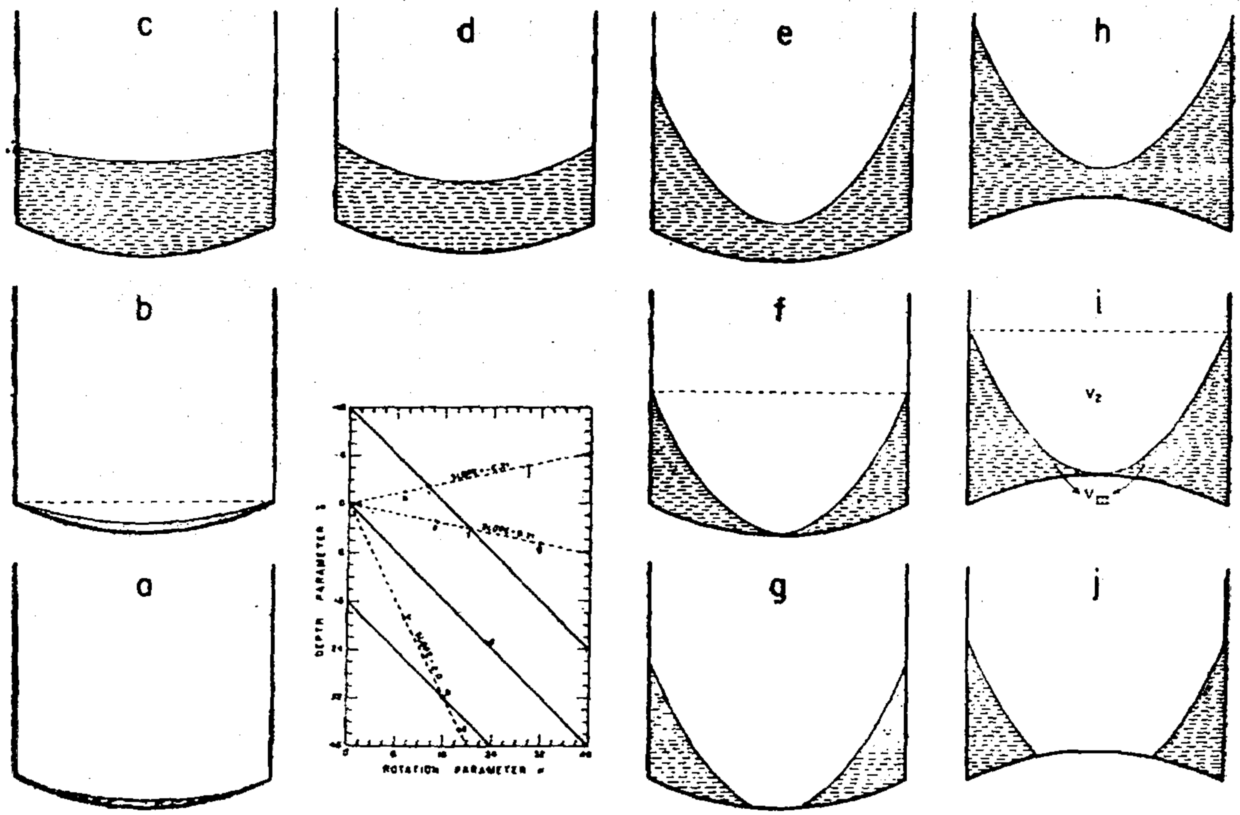

Fig. 1. The classification of the problem with respect to δ and α shown schematically. A) parabola problem, B) transition from parabola to lens problem, C) lens problem for $f > 0$, depth decreasing outward, D) lens problem for $f > 0$, depth increasing outward, F) transition from lens to annulus problem for $f > 0$, L) lens problem for $f < 0$, I) transition from lens to annulus problem for $f < 0$, J) annulus problem for $f < 0$.

$16 < \delta < \infty$ the liquid is present only in the parabola (figure 1a). The value $\delta = 0$ gives the case of a circular cylinder with flat bottom. Negative values of δ correspond to the case of a circular cylinder with a reversed paraboloidal bottom (convex to the liquid, $f < 0$). The value $\delta = -16$ gives the case in which the top of the liquid is in a horizontal level with the apex of the parabola. For all values of $\delta < -16$, the liquid level is below thte apex of the parabola and the space occupied by the liquid is doubly connected. The range of values δ takes is $-\omega \leqq \delta \leqq \infty$.

Apart from δ we have to consider the rotation parameter α. Depending on δ and α, we can distinguish broadly among the following three cases.

Case I: Parabola problem (figure 1a)

In this case liquid is present only in the parabola. This case is possible only if $f > 0$. In the nonrotating state, the top surface of the liquid is horizontal. As rotation starts, the liquid rises at the edges and sinks at the center. For a particular value of α denoted by α^{I}, the liquid just touches the cylinder at the edges (figure 1b). Thus, the relevant range of rotation in case 1 is $0 \leq \alpha \leq \alpha^{I}$. As mentioned already, for the parabola problem $\delta \geq 16$.

Case II: Lens case (figures 1c, 1d, 1e, and 1h)

In this case liquid is in contact with both the cylinder and the parabola. This case is possible for both $f > 0$ and $f < 0$. The region occupied by the liquid is of the shape of a lens. As rotation increases, the thickness of the liquid film decreases at the center and increases at the edges. For a particular value of α denoted by α^{III} the thickness of the liquid film at the center is zero, in other words the liquid touches the bottom on the axis (figures 1f and 1i). Hence for case II for $-16 < \delta < 16$, the rotation range is $0 \leq \alpha \leq \alpha^{III}$ and for $\delta \geq 16$ the rotation range is $\alpha^{I} < \alpha < \alpha^{III}$.

Case III: Annulus case (figures 1g and 1j)

In this case also the liquid is in contact with both the cylinder and the parabola. The rotations are such that the liquid is present in an annulus, there being no liquid on the axis. This case is possible both for $f > 0$ and $f < 0$. For the range $-\infty < \delta - 16$, there is no α^{III} and even in the nonrotating state the liquid is in an annulus. Hence for this case, that is for $-\infty < \delta < -16$ the rotation range is $0 < \alpha < \infty$ and for $-16 < \delta < \infty$ the rotation range is $\alpha^{III} < \alpha < \infty$ As $\alpha \to \infty$ the liquid forms an infinitely long and infinitesimally thin strip parallel to the walls.

These three cases are shown in figure 1 inset along with the lines $\alpha = \alpha^{I}$ and $\alpha = \alpha^{III}$ in the $\delta - \alpha$ plane with δ as the ordinate and α as the abcissa. The line $\alpha = \alpha^{I}$ separates regions I and II. The line $\alpha = \alpha^{III}$ separates regions II and III. The region to the left of the line $\alpha = \alpha^{I}$ represents the parabola problem (case 1). The region between the lines $\alpha = \alpha^{I}$ and $\alpha = \alpha^{III}$ represents the lens case. The region to the right of the line $\alpha = \alpha^{III}$ represents the annulus case. The line labeled $\alpha = \alpha_{eq}$ (α - equilibrium) is not a separating line between two regions but is still significant. This line exists only in that portion of the δ-α plane where $\delta \geq 0$. That is, α_{eq} is relevant only for $f > 0$. For a given δ, for $\alpha = \alpha_{eq}$ the depth of the liquid is uniform from the center to the edge. The expression for α_{eq} will be developed later. Using the definitions of δ and α from (1.7) and (1.3) respectively we get

$$\frac{\delta}{\alpha} = \frac{g}{2f\omega^2} \tag{3.1}$$

To express δ/α in terms of the focal lengths of the free surface and the bottom, we need the expression for the slope of the surface during rotation.

Let us derive an expression for the slope of the free surface during rotation. Let the system containing the parabola and cylinder joined together and filled with liquid to some depth be rotated with an angular velocity ω about the vertical axis through the center. The forces are the gravity force acting downwards, the centrifugal force acting away from the center, and the pressure-gradient force which balances the resultant of these two (figure 2). Any particle on the free surface is in equilibrium under the action of these

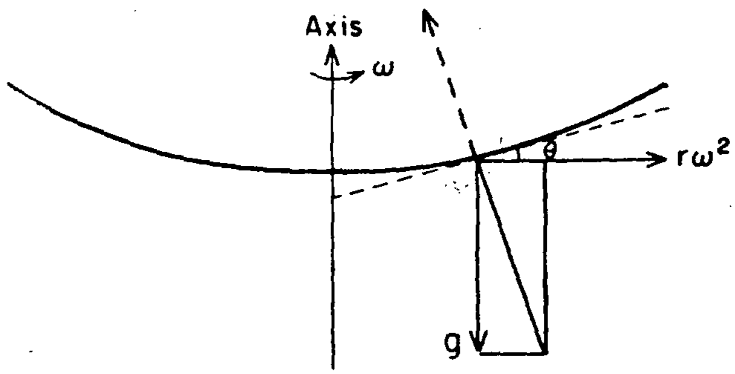

Fig. 2. The free surface of the liquid during rotation.

forces. The free surface assumes a paraboloidal form as shown in figure 2. Taking the components tangential to the surface

$$r\,\omega^2 \cos\theta = g \sin\theta$$

$$\tan\theta = \frac{r\,\omega^2}{g} \qquad (3.2)$$

Equation (3.2) gives the slope dz/dr of the free surface during rotation. Integrating (3.2) from 0 to a with respect to r we get

$$\Delta = \frac{\omega^2\ a^2}{2g} \qquad (3.3)$$

where Δ is the center depth of the free surface parabola. Also we have the relation $\Delta = a^2/4\hat{f}$ where $\hat{f}$ is the focal length of the free surface parabola. Using this result in conjunction with (3.3) in (3.1), we get

$$\frac{\delta}{\alpha} = \frac{\hat{f}}{f} \qquad (3.4)$$

Here f is the focal length of the paraboloidal bottom (f = 24 cm). For figures 1a, 1b and1c, $\hat{f}$ is taken as 48 cm, so that the slope of the δ/α line is 2. In figure 1 inset these cases are shown along with the other cases. The case 1a is the parabola problem and it lies in region I. The case 1b is the transition between the parabola case and the lens case and hence lies on the line $\alpha = \alpha_I$. The case 1c is one of positive δ in which the depth of the liquid is greatest at the center and decreases towards the edges. This situation is not possible for $f < 0$. Hence there is no point corresponding to c in the region of negative δ. The case 1d is that of uniform depth and as mentioned above is possible only for $f > 0$. The case 1e is one in which the depth of the liquid is least at the center and increases towards the edges. This case is possible for $f < 0$ also. Thus the corresponding case to 1e is 1h. Cases 1f ($f > 0$) and 1i ($f < 0$) are those in which the free surface touches the bottom at the center. This happens for $\alpha = \alpha^{III}$. Cases 1f ($f > 0$) and 1i ($f < 0$) are those in which the free surface touches the bottom at the center. This happens for $\alpha = \alpha^{III}$ Cases 1g and 1j are the annulus cases for $f > 0$ and $f < 0$ respectively. For figures 1e to 1j, $\hat{f}$ is taken at 5 cm so that the slope of the δ / α line is 0.21.

Let us derive expressions for α^{I} and α^{III} in terms of the known quantities. To derive an expression for α^{I} consider figure 1b. Let ω^{I} denote the value of ω which makes the liquid approach the bottom of the cylinder. Using (3.3) we get

$$\Delta = \int_{0}^{a} \frac{dz}{dr} \, dr = \frac{\omega^{I^2} a^2}{2g} \qquad (3.5)$$

where Δ is the depth of the free surface paraboloid (the vertical distance between the top and bottom of the paraboloid measured along the axis.) Let V_I be the volume of the liquid and V_2 be the volume indicated in figure 1b. The sum of the volumes V_I and V_2 is V_3. From the conservation of volume we can write

$$V_I = V_3 - V_2 = \frac{\pi a^4}{8f} - \frac{\pi \omega^{I^2} a^4}{4g}$$

Using the definitions of α H and δ from (1.3), (1.5) and (1.6), we get

$$\alpha^{I} \equiv \delta - 16 \qquad (3.6)$$

Since α can never be negative, from (3.6) it can be seen that α^{I} is relevant only for values of $\delta \geq 16$ (parabola problem).

To derive an expression for α^{III}, consider figures 1f and 1i. From (3.3) using a similar reasoning, we get

$$\Delta = \omega^{III^2} a^2 / 2g$$

The volume of the liquid is given by $V_{III} = V_3 - V_2$. However the expressions for V_2 and V_3 are different for this case. Using the appropriate expressions for V_2 and V_3, we get

$$V_{III} = \left\{ \frac{\pi a^4}{8f} - \pi a^2 \left(\Delta - \frac{a^2}{4f}\right) \right\} - \frac{\pi \omega_{III}^2 a^4}{4g}$$

In this using the definitions of α, δ and H, we get

$$\alpha^{III} \equiv \delta + 16 \tag{3.7}$$

Let us derive the expression for h* for case I. With reference to figure 1a, let z_0 and z_1 be the equations of the bottom and the free surface during rotation respectively with origin for z at the apex of the parabola. We can write

$$z_o = r^2/4f \tag{3.8}$$

and

$$z_1 = A+B\ r^2 \tag{3.9}$$

where A and B are constant to be determined. From (3.2), we have the slope of the free surface $\frac{dz_1}{dr} = \frac{\omega^2 r}{g}$, hence $B = \omega^2/2g$ in (3.9) and the law of depth $h = z_1 - z_o$ can be written as

$$h = A + \left(\frac{w^2}{2g} - \frac{1}{4f}\right) r^2 \tag{3.10}$$

Let $r = R$ be the radius at the outer edge of the liquid where $h = 0$. Then (3.10) can be written

$$h = \left(\frac{1}{4f} - \frac{\omega^2}{2g}\right) R^2 \ (1-\rho^2)$$

$$\rho \equiv r/R \tag{3.11}$$

The next step is to determine R^2 in terms of known quantities. The volume of the liquid is given by $V = 2\pi \int_o^R h\,r\,d\,r$, so from (3.11), we get

$$R^2 \equiv \left[\frac{2V}{\pi\left(\frac{1}{4f} - \frac{\omega^2}{2g}\right)} \right]^{1/2} \tag{3.12}$$

Using the definitions of α, δ and H from (1.3), (1.5) and (1.6), (3.12) can be written as

$$R^2 = r\,a^2/\sqrt{t} \qquad (3.13)$$
$$t \equiv \delta - \alpha$$

Writing $h^* \equiv h/H$ in the standard form $p + q\ \rho^2$ we get from (3.11)

$$p = -q = \tfrac{1}{2}\sqrt{t} \qquad (3.14)$$

The concept of $_{\alpha}$I is already given. It is that value of α which makes $R = a$. Using this in (3.13), we get the same expression for $_{\alpha}$I already given by (3.6).

Next let us derive the expression for h^* for case II. As for case I, the law of depth, written in general terms, is given by (3.10). The constant A may be expressed in terms of the volume of the liquid $V = 2\pi \int_0^a hr\,dr$, this gives

$$h = \frac{V}{\pi a^2} + \left(\frac{\omega^2}{2g} - \frac{1}{4f}\right)\left(r^2 - \tfrac{1}{2}a^2\right)$$

Using the definitions of h^* and ρ from (2.11), δ, α and H from (1.3), (1.5) and (1.6), we get

$$h^* = p + q\,\rho^2$$
$$p = 1 + \frac{t}{16} \qquad (3.15)$$
$$q = -\frac{t}{8}$$

The concept of $_{\alpha}$III is already given. It is that value of α for which the liquid just touches the bottom at the center. That is $h^* = 0$ for $\rho = 0$. This happens for $p = 0$. From (3.15) we get $t = -16$ and from (3.13) we get $_{\alpha}\text{III} = \delta + 16$ which is the same result obtained previously and given by (3.7). For a particular value of α called α-equilibrium (α_{eq}) the liquid has uniform depth. That is h^* is independent of ρ. This happens if $q = 0$. By putting $q = 0$ we get from (3.15) and (3.13)

$$\alpha_{eq} \equiv \delta \qquad (3.16)$$

The parameter α_{eq} is relevant only for cases with positive values of δ ($f > 0$).

Next consider the ring case III. The law of depth in general terms is given by (3.10). Let $r = r_0$ be the radius at the inner edge of the liquid where $h = 0$. Then (3.10) can be written as

$$h = \left(\frac{\omega^2 a^2}{2g} - \frac{a^2}{4f}\right)\left(\rho^2 - \rho_0^2\right) \qquad (3.17)$$

$$\rho_o = \frac{r_o}{a}$$

The next step is to determine ρ_o in terms of known quantities. The volume of the liquid is given by $V = 2\pi \int_{r_o}^{a}$

$$\rho_o = 1 - \left[\frac{8\, fV}{\pi a^4 \left(\frac{2\,\omega^2 f}{g} - 1 \right)} \right]^{1/2}$$

Using the definitions of α, δ and H from (1.3), (1.5) and (1.6) we can write

$$\rho_o^2 = 1 - \frac{4}{\sqrt{-t}} \tag{3.18}$$

From (3.17) and (3.18) we get

$$\left.\begin{aligned} h^* &= p + q\ \rho^2 \\ p &= \frac{t}{8} + \frac{1}{2}\sqrt{-t} \\ q &= -\frac{t}{8} \end{aligned}\right\} \tag{3.19}$$

when $\rho_o = 0$ we have $\alpha = \alpha_{III}$ From (3.18) putting $\rho_o = 0$ we get $t = -16$ or $\alpha_{III} = \delta + 16$ which is the same result already derived.

4. LEGENDRE FUNCTION SOLUTIONS

In the third section, the various cases of the present problem have been formulated and the expressions for h* are derived. In section 2, the solution of the general problem (circular cylinder with arbitrary law of depth) is given, making use of the Galerkin method. In this section, it will be shown that the parabola-cylinder problem (not the general problem) can be solved using the Legendre functions. The question naturally arises: when the solution is available in a closed form, why use the Galerkin method at all which gives only open form solutions? The answer is that the Legendre functions are not tabulated very extensively and so from a practical point of view we still need the Galerkin method through which we can readily calculate the frequencies for arbitrary combinations of the depth and the rotation parameters.

The differential equation for the rotating axisymmetric modes can be written from (2.11)

$$\frac{d}{d\rho}\left(h^*\rho\ \frac{d\eta}{d\rho}\right) + (\beta^2 - \alpha)\,\rho\,\eta = 0$$

$$h^* = p + q\ \rho^2 \tag{4.1}$$

Making a transformation of the independent variable from ρ to y where

$$y = 1 + \frac{2q}{p}\,\rho^2 \quad , \tag{4.2}$$

equation (4.1) reduces to

$$\frac{d}{dy}\left[(1-y^2)\frac{d\eta}{dy}\right]+n(n=1)\eta=0 \quad (4.3)$$

where

$$n(n+1) \equiv -\frac{(\beta^2-\alpha)}{4q} \quad (4.4)$$

Equation (4.3) is the standard form of the equation for the Legendre functions. For all the cases, the dimensionless depth h^* can be expressed in the form $p+q\ \rho^2$. Hence for all the cases the Sturm-Liouville equation can be reduced to the standard form of the Legendre equation given by (4.3). The solution of this is a linear combination of $P_n(y)$ and $Q_n(y)$ namely

$$\eta = BP_n(y)+FQ_n(y) \quad (4.5)$$

where P_n and Q_n are the Legendre functions of the first and the second kind for arbitrary real values of n. The function $P_n(y)$ is singular at $y=-1$ unless n is an integer and is always singular at $y=\infty$. The function $Q_n(y)$ is singular at $y=\neq 1$ and also at $y=\infty$. Define

$$Y \equiv \frac{1+2q}{p} \quad (4.6)$$

Then (4.5) can be written as

$$\eta = E\,P_n\left\{1+(Y-1)\ \rho^2\right\} + F\,Q_n\left\{1+(Y-1)\,\rho^2\right\} \quad (4.7)$$

Next let us discuss the boundary conditions of the various cases. For case I, the boundary condition at the outer edge is

$$\eta)_{\rho=1} \neq \infty \quad (4.8)$$

This means at the outer edge of the liquid we demand finiteness of η The boundary condition at the center is

$$\frac{d\eta}{d\rho}\Big)_{=0}=0$$

This is because the modes are axisymmetric. For cases II and III, the boundary condition at the outer wall is

$$\frac{d\eta}{d\rho}\Big)_{\rho=1}=0 \quad (4.9)$$

This is because the walls are vertical. The boundary condition at the center for case II is $\frac{d\eta}{d\rho}\Big)_{\rho=0} = 0$ as for case I. However for case III, the inner boundary condition is

$$\eta)_{\rho=\rho_0} \neq \infty \tag{4.10}$$

Because of the different boundary conditions, we have different types of solution for the various cases.

The relevant range of ρ is from 0 to 1. Using (4.2) and (4.6) we get

$$1 \leqq y \leqq Y \text{ if } Y > 1$$

or (4.11)

$$Y \leqq y \leqq 1 \text{ if } Y < 1$$

At the outer wall, since $\rho = 1$ we have $y = Y$.

First let us consider the parabola problem. From (3.14), we have $p = -q$. Using this in (4.6), we get $Y = -1$. Making use of the boundary condition (4.8) and remembering that the only solution for the Legendre differential equation given by (4.3) which is regular at both ± 1 is the Legendre polynomial $P_n(y)$ with $n = 0, 1, 2, 3, \ldots$. The frequency equation for the parabola problem can be written now. For this, start with the differential equatio (2.10). Multiplying (2.10) by R^2/g and using (3.11), we get

$$h \left(\frac{d^2\eta}{d\rho^2} + \frac{1}{\rho}\frac{d\eta}{d\rho}\right) + \frac{dh}{d\rho}\frac{d\eta}{d\rho} + (\sigma^2 - 4\omega^2)\frac{R^2}{g}\eta = 0$$

Nondimensionalizing this equation through division by H and using the definitions of β^2 and α from (1.2) and (1.3) respectively, we get

$$\frac{d}{d\rho}\left(h^* \rho \frac{d\eta}{d\rho}\right) = (\beta^2 - \alpha)\frac{R^2}{a^2} = \rho\, \eta = 0 \tag{4.12}$$

This is in the standard form of the Sturm-Liouville equation and can be converted into the Legendre equation by making the same transformation as before given by (4.2). Then the frequency equation can be written as

$$n(n+1) = -\frac{(\beta^2 - \alpha)}{4q}\frac{R^2}{a^2} \tag{4.13}$$

For this case q is given by (3.14), and R^2 is given by (3.13). Using the relation $t = \delta - \alpha$, we get

$$\beta^2 = \alpha + \tfrac{1}{2}n(n+1)(\delta - \alpha)$$

$$n = 1,2,3,\ldots \tag{4.14}$$

By putting $\alpha = 0$ in (4.14), we get the frequency equation for the nonrotating case of the parabola problem:

$$\beta_0^2 = \tfrac{1}{2}\, n(n+1)\,\delta \tag{4.15}$$

where the subscript o denotes nonrotation. Using the definition of K from (1.4), we get

$$K = 1 - \tfrac{1}{2}n(n+1) \tag{4.16}$$

The values $n = 1, 2, 3, \ldots$ determine the various modes. Thus the frequency spectrum is discrete. The above result given by (4.16) is astonishing in a sense because for the fundamental mode, rotation has no effect on the frequencies. This is in contradiction to the results obtained by Poincare who treated the parabola problem without taking into account the ellipticity of the free surface during rotation.

Next, let us consider the parabola-cylinder problem. From (4.5) and (4.9) the outer boundary condition for this case can be written as

$$E P_n'(Y) + Q_n'(Y) = 0 \tag{4.17}$$

By prescribing values for δ and α, we fix Y through p and q. Let $n_1, n_2 \ldots$ denote the roots of (4.17). The corresponding frequencies are then found from (4.4) which can be written as

$$\beta^2 = \alpha - 4q\, n\,(n+1) \tag{4.18}$$

There is a discrete set of values for β since (4.17) has only a discrete set of roots. Thus the frequency spectrum is discrete. At $\rho = 0$, we have $y = 1$ For the lens case at the center, we demand finiteness of η. From (4.5), it can be seen that η is a linear combination of $P_n(y)$ and $Q_n(y)$. But at $y = 1$, $Q_n(y)$ is singular as such we have to reject this and take only $P_n(y)$. From (4.17), the outer boundary condition for the lens case can be written as $P_n'(Y) = 0$.

Carrus and Treuenfels (1950-52) have tabulated the first 50 zeros of $P_n'(Y)$ for $Y = \cos\theta$ with $\theta = 90°(5°)\ 175$. Their table covers the range $-1 \leq Y \leq 0$. This range of Y does not cover all the cases physically possible. From the above considerations, the writers decided to use the Galerkin method described in section 2 rather than attempt a direct computation of the roots of (4.17) over the entire range of Y. However there exists a particular relation between δ and α for which the Legendre functions reduce to polynomials and therefore the problem may be solved in an elementary fashion. This happens for $Y = 0$ since in that case $P_n'(0) = 0$ is satisfied for $n = 0, 2, 4, 6, \ldots$.

The next step is to determine the relation between δ and α which gives $Y = 0$ for this case. Using the values of p and q from (3.15) and also the relation $t = \delta - \alpha$, we get for the lens case

$$\alpha = \delta + 16\,\frac{Y-1}{Y+3} \tag{4.19}$$

Putting $Y = 0$ in this, we get

$$\alpha = \bar{\alpha} \equiv \delta - \frac{16}{3} \tag{4.20}$$

For a given value of δ there are polynomial solutions for $\alpha = \bar{\alpha}$ Since $\bar{\alpha}$ cannot be negative, it follows that polynomial solutions exist only for values of δ in the range $\frac{16}{3} \leq \delta \leq 16$. Hence there are no polynomial solutions for the reversed parabola-cylinder problem, flat-bottom case and the parabola-cylinder problem for values of $\delta < \frac{16}{3}$ For the parabola problem, $Y = -1$ always and so Y can never be zero. For the annulus case, t and hence α cannot be expressed in terms of Y by means of a simple relation since we get a quadratic equation in t.

Let us write down the frequency equation for the polynomial cases. The general frequency equation is given by (4.18). Substituting $q = -\frac{t}{8}$ we get

$$\beta^2 = \bar{\alpha} + \frac{1}{2} n(n+1)(\delta - \bar{\alpha})$$

for the lens case making use of (4.20) the frequency equation can be written

$$\beta^2 = \delta - \frac{16}{3} + \frac{8}{3} n(n+1) \qquad (4.21)$$

However K cannot be calculated for these cases since β_0^2 cannot be obtained for the same value of δ by means of polynomial solutions. When polynomial solutions exist the general solution

$$\eta = E P_n[1 + (Y-1)\rho^2]$$

reduces to the polynomials $P_n(1-\rho^2)$.

Let us discuss certain general properties of the solution $P_n(y)$ using the relation between δ and α derived already. The relation between δ and α for different values of Y is plotted in the figure 3, with δ as ordinate and α as abcissa. The relevant domain of the $\delta - \alpha$ plane is $-\infty < \delta < \infty$ and $0 \leq \alpha < \infty$. The corresponding relevant values of Y are $1 < Y < \infty$. For the parabola problem $Y = -1$ always. In the $\delta - \alpha$ plane though a curve can be drawn for every value of Y lying in the above range, only for particular curves there is significance. As is already pointed out, there are polynomial solutions $P_n(1-\rho^2)$ along the curve $Y = 0$. On the curve $Y = 1$, $q = 0$. So the transformation (4.2) leading to the Legendre's equation is not valid and to treat this particular case, we have to start with the basic differenlial equation (2.10). It can be seen that $q = 0$ is relevant only for positive values of δ. However, it is not valid for the annulus case since there is no liquid on the axis and there cannot be uniform depth. For the lens case from (3.15), we have $p = 1 - \frac{1}{2}q$. Solving this simultaneously with the relation $Y = 1 + \frac{2q}{p}$ for p in terms of Y, we get

$$p = \frac{4}{Y+3} \qquad (4.22)$$

Putting $Y = 1$ this gives $p = 1$. The differential equation (2.10) after substituting $p = 1$ and $q = 0$ becomes

$$\frac{d^2\eta}{d\rho^2} + \frac{1}{\rho}\frac{d\eta}{d\rho} + (\beta^2 - \alpha_{eq})\eta = 0$$

This is the standard form of the Bessel differential equation and the solution

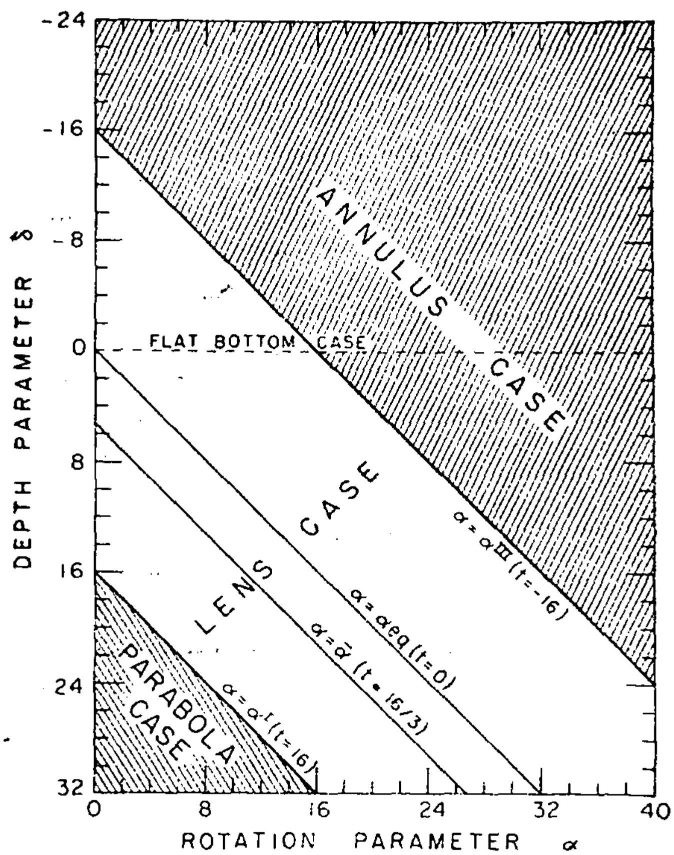

Fig. 3. The δ–α plane with the important lines shown on it. $\alpha=\alpha I$ marks the transition between the parabola problem and the lens problem. $\alpha=\alpha III$ marks the transition between the lens problem and the annulus problem. The line $\alpha=\alpha_{eq}$ corresponds to uniform depth. Along the line $\alpha=\bar{\alpha}$ there are polynomial solutions.

is in terms of the J_0 function, the Y_0 function being excluded because of its singularity at the origin. Along the curve $Y=\infty$, $p=0$. When $p=0$, $h^*=0$ at $\rho=0$ and $\alpha=\chi III$

It has been mentioned above that though the parabola-cylinder problem can be solved in a closed form, from a practical point of view the Galerkin method is selected. Now the problem is this. What functions $f_n(\rho)$ are most suitable for the application of the Galerkin method? In the above arguments, it was mentioned that there are closed form solutions along the curve $Y=0$ ($\alpha=\bar{\alpha}$ and along the curve $Y=1$ ($\alpha=\alpha_{eq}$). The former gives the Legendre polynomials $P_n(1-\rho^2)$ and the latter gives the Bessel function solutions $J_0(kn_\rho)$. In the vicinity of α the P_n functions are fast converging and in the vicinity of α_{eq} the J_0 function are preferable. But the J_0 functions are selected because the curve for α_{eq} is more centered in the δ-α plane than the curve for $\bar{\alpha}$ Also the matrix elments can be evaluated more easily if these functions are employed.

5. CONSTRUCTION OF THE ORTHO-NORMAL FUNCTIONS

In section 2, the scheme for the solution of the general problem (circular cylinder with arbitrary law of depth) is given making use of the Galerkin method. There it was mentioned that h^* is different for the different cases. In section 3, the present problem is broadly classified into three classes, namely the parabola case, the lens case and the annulus case, and the expressions for h^* for the various cases are derived.

In section 4, it was mentioned that either the Legendre P_n functions or the Bessel J_o functions are selected. The criteria for choosing the f_n function is that it should be orthogonal with the weight function ρ in the interval $0 \leqq \rho \leqq 1$ and it should satisfy the boundary conditions of the problem.

For the parabola problem, we don't need the Galerkin method because, there are polynomial solutions. It has already been stated that the Bessel differential equation of zero order has two solutions, namely J_o and Y_o. The second solution Y_o is singular at the origin. For the lens case, since there is liquid on the axis, the boundary condition requires $\zeta)_{\rho=o} \neq \infty$. Hence Y_o has to be excluded. However for the annulus case, there is no reason to exclude Y_o and the f_n function should be chosen differently. We have not considered this case. Thus we are left with the lens case only. For this case the f_n function is chosen as

$$f_n(\rho) \equiv \sqrt{2}\ \frac{J_o(k_n \rho)}{J_o(k_n} \qquad (5.1)$$

with a similar expression for $f_m(\rho)$. These f_n functions form an orthonormal set and they individually satisfy the boundary conditions of the problem. Here k_n, $n = 1, 2, 3 \ldots$ are the roots of $J_1 \ (k_n) = 0$.

Using the relation $h^* = p + q \ \ \rho^2$ in conjunction with (2.17) and (5.1) we get the following expression for the matrix elements.

$$m/n = p\, k_m k_n \int_0^1 g_m\, g_n\, \rho\, d\rho + q\, k_m k_n \int_0^1 g_m\, g_n\, \rho^3\, d\rho \qquad (5.2)$$

$$g_n(\rho) \equiv \sqrt{2}\,\frac{-J_1(k_n \rho)}{J_o(k_n)} \qquad (5.3)$$

To calculate the matrix elements, we have to evaluate two Lommel integrals that appear in (5.2). The first one present in the first term on the right hand side is a standard integral. Its value is from Hidaka (1931)

$$\int_0^1 g_m\, g_n\, \rho\, d\rho = \delta_{m,n} \qquad (5.4)$$

where

$$\delta_{m,n} = 1 \text{ for } m = n$$
$$= 0 \text{ for } m \neq n$$

The second integral is evaluated by Platzman (1962)

$$\int_0^1 g_m\, g_n\, \rho^3\, d\rho = \tfrac{1}{3} \quad \text{for } m = n$$

$$= 8\, k_m k_n / (k^2_m - k^2_n)^2 \qquad \text{for } m \neq n \tag{5.5}$$

Substituting (5.4) and (5.5) in (5.2) we get

$$(n/n) = k^2_n\, (p + \tfrac{1}{3} q)$$

$$(m/n) = 8q \frac{k^2_m\, k^2_n}{(k^2_m - k^2_n)^2} \tag{5.6}$$

The next step is to write the matrix coefficients explicitely. From (3.15) it can be written

$$(n/n) = k^2_n \left(1 + \frac{t}{48}\right)$$

$$(m/n) = \frac{t\, k^4_m\, k^4_n}{(k^2_m - k^2_n)^2} \tag{5.7}$$

The successive approximations to $\beta^2-\alpha$ are made as follows. The first approximation is made by setting

$$\left| (\beta^2-\alpha) - A_{11} \right| = 0$$

where

$$A_{mn} = (m/n)$$

This gives $\beta^2-\alpha$ for the (0.1) mode under the first approximation. The second approximation is made by setting

$$\begin{vmatrix} (\beta^2-\alpha) - A_{11} & A_{12} \\ A_{21} & (\beta^2-\alpha) - A_{22} \end{vmatrix} = 0$$

This gives a quadratic equation in $\beta^2-\alpha$. The lowest value corresponds to the (0,1) mode and the other value to the (0,2) mode. In a similar way an approximation of any desired order can be made and the frequencies of the various modes can be determined.

6. NUMERICAL CALCULATION AND DISCUSSION OF THE RESULTS

In section 5, we have obtained the expressions for the matrix elements. The characteristic equation for the frequency is

$$\det \{\lambda \delta_{m,n} - A_{mn}\} = 0 \tag{6.1}$$

where

$$\lambda \equiv \beta^2 - \alpha$$

$$\delta_{m,n} = 1 \text{ for } m = n$$

$$= 0 \text{ for } m \neq n$$

Also we have

$$A_{mn} = k_n^2 \left(1 + \frac{t}{48}\right)$$

$$A_{mn} = - t\, k^2_m k^2_n / (k^2_m - k^2_n)^2 \tag{6.2}$$

Here k_n and k_m are the n th and the m th zeros of J_1 (k). The problem is to calculate the frequency-parameter K as a function of δ and α for different modes under a suitable approximation. To calculate K, we need the eigenvalues λ.

This is a characteristic value problem and the characteristic values we seek are the eigenvalues of an NxN symmetric matrix. Solving an NxN matrix means making the N th approximation. If A is a square symmetric matrix, the problem is to find a number λ and a vector $\vec{V}$ such that

$$A\vec{V} = \lambda\vec{V} \tag{6.3}$$

Each λ is referred to as an eigenvalue of the matrix A and each related N-dimensional vector $\vec{V}$ is called an eigenvector corresponding to the eigenvalue. The eigenvectors are mutually orthogonal. Solution exists if and only if

$$\det (A - \lambda\, II) = 0$$

where II is an N th order unit matrix. This relationship results in an N th degree polynomial which has in general N real roots.

The initial calculations are made using the Jacobi method. In the Jacobi method, a square symmetric matrix is reduced to the diagonal form by a series of plane rotations. In this process the off-diagonal elements are made smaller than a prescribed value Δ. If Δ is too small or too large, precision will be lost. In the calculations Δ is chosen as 10^{-6}. The results of these calculations were not at all encouraging. At first, the frequencies and K values were calculated under the 5th and the 6th approximations. The results of K were inconsistent in the following sense. Since this is a self-adjoint problem amenable to variational methods, the frequency and hence K values calculated under a higher approximation should be less than the corresponding values calculated under a lower approximation. But in most of the cases calculated, K for a higher approximation was greater than that for a lower approximation. But this inconsistancy was not shown by the frequency values. Then if the frequency values were alright, why were the K values wrong? The answer was that, K is given by $K = \frac{\beta^2 - \beta_o^2}{\alpha}$ The β_o^2 values should be essentially the same for the two approximations, since here we were dealing with the small difference between two large quantities. It was found that β_o^2 values were not converging fast enough for the 5th and the 6th approximations. So the calculations were repeated for the 9th and the 10th approximations. However β_o^2 under the 10th approximation was used to calculate the K values even under the 9th approximation. When this was done, the discrepancy disappeared among the K values. Even under the 10th approximation the results were not satisfactory. This was judged by comparison with a few cases for which exact values were known by other methods. Since the program was carried on single precision, the number of digits is eight. The eigenvalues calculated were also lower than what they should be.

Since the initial results were not accurate enough the calculations are repeated using a method different from Jacobi's. The method used is the Givens (1954) method. This has some advantages over the Jacobi method. In this method the nondiagonal elements are exactly reduced to zero. In these calculations, double precision is used so that the number of digits is 16.

The eigenvalues are calculated from $t = -16$ to $t = 16$ in steps of unity. This range covers the lens case completely. In addition to these values, eigenvalues are calculated for $t = 14.36, 7.30, 3.65$ and 2.4. From (1.4) we have

$$K \equiv \frac{\beta^2 - \beta_o^2}{\alpha}$$

From (3.10) we have $t = \delta - \alpha$. From (6.1) we have $\lambda = \beta^2 - \alpha$. Using these it can be written

$$K = 1 + \left(\frac{\lambda_{\delta-\alpha} - \lambda_{\delta}}{\alpha} \right) \tag{6.4}$$

The range of values of δ for the lens case is $-16 \leq \delta \leq 16$. First a particular value of δ is chosen. Then for all the t values for which the eigenvalues are calculated, for this δ the corresponding α values are calculated from the relation $\alpha = \delta - t$ and then K is calculated from (6.4). However using this formula K can be calculated only for $\alpha > 0$.

To calculate $K_{\alpha \to 0}$ we need a different method. From (6.3) we have

$$A\vec{V} = \lambda\vec{V} \tag{6.5}$$

where

$$\vec{V} = \text{column } (V_1, V_2, V_3, \ldots)$$

Write

$$A_{mn} = B_{mn} + \alpha\, C_{mn} \tag{6.6}$$

Let us write

$$\lambda = \lambda_o + \alpha\, \lambda_1 = O(\alpha^2) \tag{6.7}$$

and

$$\vec{V} = \vec{V}_o + \alpha\, \vec{V}_1 + O(\alpha^2) \tag{6.8}$$

Substitute (6.6), (6.7) and (6.8) in (6.3) to give

$$(\overleftrightarrow{B} + \alpha\, C)\, (\vec{V}_o + \vec{V}_1) = (\lambda_o + \alpha\, \lambda_1)\, (\overleftrightarrow{V}_o + \alpha\, \vec{V}_1) + O(\alpha^2)$$

Neglect the terms of $O(\alpha^2)$ in this equation. Also noting that

$$\overleftrightarrow{B}\, \vec{V}_o = \lambda_o\, \vec{V}_o$$

we get

$$\lambda_1 \vec{V}_o = C\vec{V}_o + (B - \lambda_o\, II)\, \overleftrightarrow{V}_1$$

But $(B - \lambda_o\, II)\, V_1$ is orthogonal to $\vec{V}_o$; therefore $\vec{V}_o^{\,T}$ as a prefactor annihilates this term, and we get

$$\lambda_1 \vec{V}_o^{\,T}\, \overleftrightarrow{V}_o = \vec{V}_o^{\,T}\, C\vec{V}_o$$

since $\vec{V}_o^{\,T}\, \vec{V}_o = 1$ this reduces to

$$\lambda_1 = \overleftrightarrow{V}_o^{\,T}\, C\overleftrightarrow{V}_o \tag{6.9}$$

From (6.1) we have

$$\lambda \equiv \beta^2 - \alpha$$

Using (6.7) we get $\beta_o^2 = \lambda_o$ and

$$K = 1 + \lambda_1 \tag{6.10}$$

The elements of C can be written after making use of (6.1) and (6.6)

$$C_{nn} = -k_n^2/48$$
$$C_{mn} = \frac{k_m^2\ k_n^2}{(k_m^2 - k_m^2)^2} \tag{6.11}$$

Making use of (6.9), (6.10) and (6.11) $K_{\alpha \to 0}$ is calculated.

The theoretical results are presented in two types of graphs (Figure 4). In the first type for a given δ value, K versus α curve is plotted for the first three modes on the same graph. Three δ values are selected for this

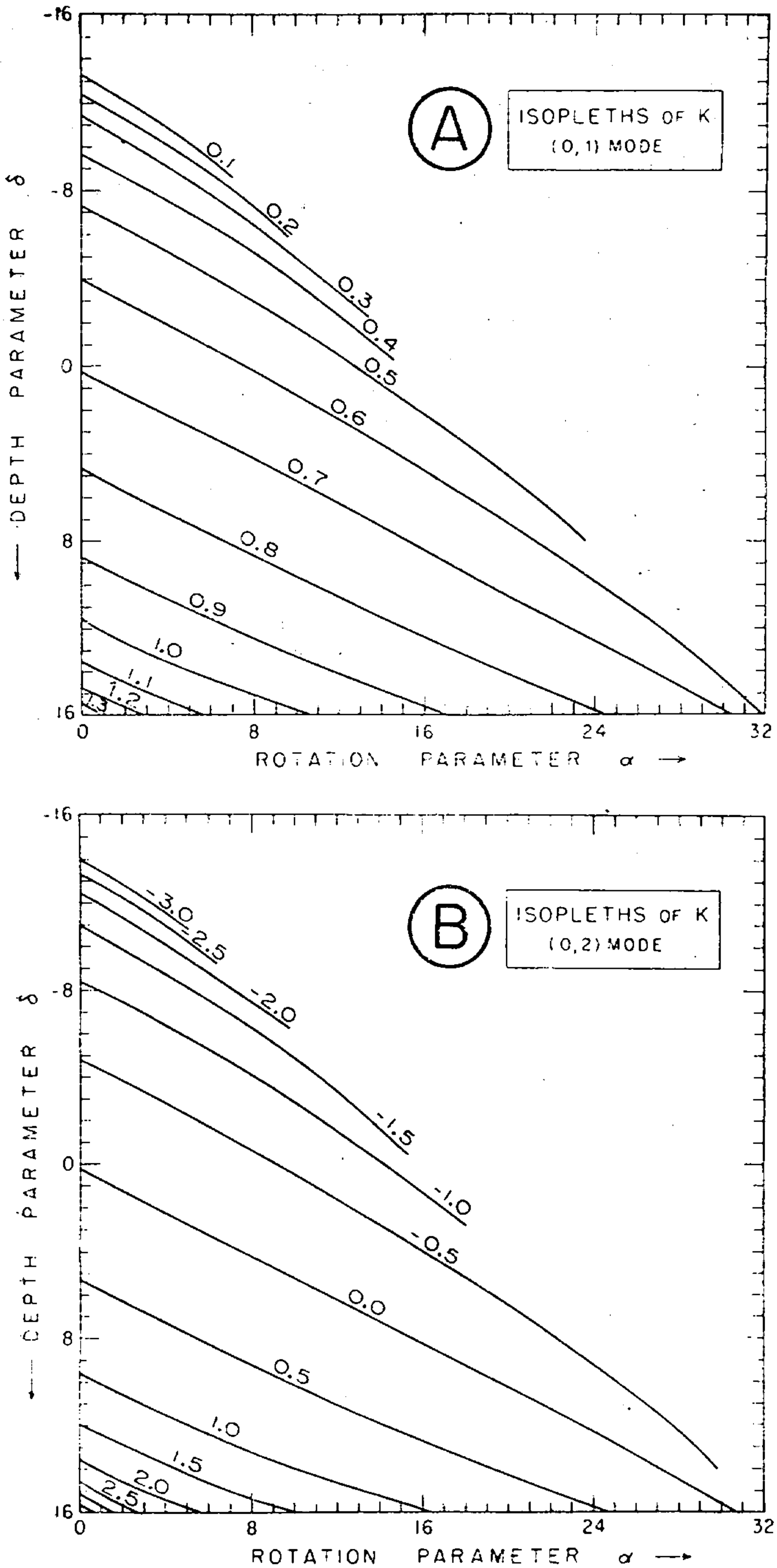

Fig. 4. A) Isopleths of K drawn in the δ–α plane using the theoretical calculations (0, 1) mode. B) Isopleths of K drawn in the δ–α plane using the theoretical calculations (0, 2) mode.

purpose. The first one is $\delta = 0$ which is the flat-bottom case. The other two are 14.36 and 7.30 which are the shallowest cases used in the experiments. In the second type of representation, the isopleths of K are drawn in the $\delta - \alpha$ plane for each mode separately. The theoretical K versus α curves show the following features. For $\delta = 0$ (flat-bottom) in the whole range of α. $K_1 > K_2 > K_3$. Hence the curve for the fundamental mode is above the curve for the second mode which in turn is above the curve for the third mode. The K_1 curve falls off rather flatly. The curve for any higher mode falls off more steeply than the curve for a lower mode. Next consider the shallowest paraboloidal bottom case ($\delta = 14.36$). For this case the K_1 curve for the $\delta = 0$ case, especially at the beginning (close to $\alpha = 0$) and at the end (close to $\alpha = \alpha^{III}$). The K_2 curve starts higher but at $\alpha \sim 13.1$ it crosses the K_1 curve and from there on stays below the K_1 curve. The K_3 curve starts above the K_2 curve and crosses it at $\alpha \sim 13.7$. Still one thing is common between this case and the previous case. The higher the mode the steeper is the curve. Next consider a less shallow case with paraboloidal bottom namely $\delta = 7.30$. For this case there are no crossings between the curves. The K_1 curve falls off flatly as usual. The K_2 curve starts lower and falls off more steeply. The K_3 curve starts still lower and falls off much more steeply.

It was mentioned already that the case $\delta = 16$ is one with $f > 0$ in which the parabola is completely filled and the liquid stands at the bottom of the cylinder. If the volume of the liquid is slightly decreased we have the parabola problem and if the volume of the liquid is slightly increased, we have the lens case. So we expect that if we approach $\delta = 16$ from either side we should get the same value of K. However it was shown in section 4 that for the parabola problem for the fundamental mode, $K = 0$. However using the solution for the lens case we get $K_1 = 1.38$ for $\delta = 16$. Obviously there appears to be a controversy. This can be explained easily because in the $\delta - \alpha$ plane there is a discontinuity along the line $\alpha = \alpha^{I}$ which marks the transition between the parabola problem and the lens problem. Because of this discontinuity, it is natural for K to have discontinuity along this line.

Next let us look at the isopleths of K in the $\delta - \alpha$ plane. The isopleths are not straight lines but have little curvature. In the region of large values of δ they have a concave curvature and as δ approaches zero the curvature decreases. For negative values of δ they have a convex curvature. Within the precision allowed by the number of calculated values, the isopleth pattern for the fundamental mode is the same as for the second mode, though the individual values are different. This fact suggests the following idea. The frequency parameter K which is a function of the depth parameter δ the rotation parameter α. and mode has only two degrees of freedom instead of three. Let ϕ be some constant on the loci of the isopleth of K. Then $\phi = \phi(\delta, \alpha)$ and $K = K(\phi \text{ m Mode})$.

Next let us compare the values obtained by the Galerkin method with those obtained by using the Legendre function solutions. First consider the case $\delta = 16$. For $\alpha = 0$ this comes under the parabola problem. For this case the frequency equation is given by (4.15) namely

$$\beta_0^2 = \tfrac{1}{2}\, n\,(n+1)\,\delta \text{ putting } \delta = 16 \text{ we get}$$

$$\beta_0^2 = 8\,n\,(n+1) \qquad n = 1, 2, 3, \ldots \tag{6.12}$$

Table I shows the β_0^2 values calculated from (6.12) side by side with those obtained by the Galerkin method.

Table I. Comparison of the approximate β_0^2 values obtained by using the Galerkin method with the exact values obtained by using the Legendre functions for $\delta = 16$.

Mode	Exact value	Approximate value	Percentage error
1	16.0	16.041156	0.257
2	48.0	48.557425	1.161
3	96.0	98.695081	2.807
4	160.0	168.08142	5.051
5	240.0	258.46557	7.694

It was shown in section 4 that for the lens case there are polynomial solutions. We have the relation $\lambda = \beta^2 - \alpha$ where λ represents the eigenvalues. The expression for $\bar{\alpha}$ is $\bar{\alpha} = \delta - \frac{16}{3}$, if we choose $\delta = \frac{16}{3}$ then $\bar{a} = 0$. Thus for $\delta = \frac{16}{3}$ for the nonrotating case there are polynomial solutions. The frequency equation for $\delta = \frac{16}{3}$ under the condition of polynomial solutions can be written from (4.21).

$$\beta^2 = \tfrac{8}{3}\, n\,(n+1)$$

$$n = 2, 4, 6 \ldots$$

Using this equation the β^2 values can be calculated. Table II shows these values side by side with those obtained by using the Galerkin method.

Table II. Comparison of the approximate β^2 values obtained by using the Galerkin method with the exact values obtained by using the Legendre functions for $\delta = \frac{16}{3}$

Modes	Exact value	Approximate value	Percentage error
1	16.0	16.000002	1.250×10^{-5}
2	53.333333	53.333359	4.875×10^{-5}
3	112.0	112.000010	8.929×10^{-5}
4	192.0	192.00035	18.23×10^{-5}
5	293.33333	293.33427	32.05×10^{-5}

In the application of the Galerkin method, the J_0 function is chosen as the f_n function. The J_0 function solves only the case $\delta = 0$ and $\alpha = 0$. So naturally for increasing values of δ and a we expect the convergence to be poor. Also the higher the mode the worse will be the values. But Tables I and II show that the Galerkin method solution is satisfactory at least for the first three modes.

In section 1, it was mentioned that rotation does not necessarily increase the frequency and hence the restoring tendency. This fact can be seen from the frequencies for the higher modes.

7. CONCLUSIONS

The effect of ellipticity of the free surface on the frequencies of gravity modes is not negligible as is treated in the classical shallow water theory. For the fundamental mode, the frequency parameter K is of the order of unity, but for higher modes the absolute value of K is large. Hence ellipticity effects the frequencies of the higher modes more profoundly than the fundamental mode. One conspicuous result is that K is a monotone decreasing function of α. It takes both positive and negative values depending upon δ, a and mode. The writers have no specific answers in a physical sense to the following two questions.

1) Why is K a monotone decreasing function of a?

2) Why should K change its sign?

To answer these questions, one may have to go into a detailed dynamics of the system, for example the dynamic pressure field, the kinetic and potential energies, etc.

Another important result brought out in this paper is that rotation has no effect on the fundamental gravity mode in a parabola. This is an unexpected result because such a result is not obtained for any other law of depth.

The theory in this paper is based on shallow water approximation and it is difficult to say how far this is appropriate for a cylinder with paraboloidal bottom. Regarding geophysical applications of the problem, this theory has applications to the tidal problems. If this theory is extended to two layers of different densities then we will have a situation comparable to the polar front problem.

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